

# Isomonodromic Deformations with an Irregular Singularity and Hyperelliptic Curves

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**Abstract:** In this paper, we extend the result of Kitaev and Korotkin [7] to the case where a monodromy-preserving deformation has an irregular singularity. For the monodromy-preserving deformation, we obtain the  $\tau$ -function whose deformation parameters are the positions of regular singularities and the parameter  $t$  of an irregular singularity. Furthermore, the  $\tau$ -function is expressed by the hyperelliptic  $\Theta$  function moving the argument  $\mathbf{z}$  and the period  $\mathbf{B}$ , where  $t$  and the positions of regular singularities move  $z$  and  $\mathbf{B}$ , respectively.

**Key words:** the  $\tau$ -function; the  $\Theta$  function; monodromy-preserving deformation; irregular singular point; hyperelliptic curves.

## Introduction

In this paper, we extend the result of Kitaev and Korotkin [7] to the case where a monodromy-preserving deformation has an irregular singularity. For the monodromy-preserving deformation, we obtain the  $\tau$ -function represented by the hyperelliptic  $\Theta$  function moving both the argument  $\mathbf{z}$  and the period  $\mathbf{B}$ . In [4], we constructed the  $\tau$ -function by the elliptic  $\Theta$  function moving the argument  $z$  and the period  $\Omega$ .

Miwa, Jimbo and Ueno [5] extended the work of Schlesinger [11] and established a general theory of monodromy-preserving deformation for a first order matrix system of ordinary linear differential equations,

$$\frac{dY}{dx} = A(x)Y, \quad A(x) = \sum_{\nu=1}^n \sum_{k=0}^{r_\nu} \frac{A_{\nu,-k}}{(x - a_\nu)^{k+1}} - \sum_{k=1}^{r_\infty} A_{\infty,-k} x^{k-1}, \quad (0.1)$$

having regular or irregular singularities of arbitrary rank.

The monodromy data to be preserved are

- (i) Stokes multipliers  $S_j^{(\nu)}$  ( $j = 1, \dots, 2r_\nu$ ),
- (ii) connection matrices  $C^{(\nu)}$ ,
- (iii) “exponents of formal monodromy”  $T_0^{(\nu)}$ .

Miwa, Jimbo and Ueno found a deformation equation as a necessary and sufficient condition for the monodromy data to be independent of deformation parameters and defined the  $\tau$ -function for the deformation equation.

Let us explain the relationship between the  $\tau$ -function and the  $\Theta$  function. Miwa and Jimbo [6] constructed a monodromy-preserving deformation with irregular singularities and expressed the  $\tau$ -function with the  $\Theta$  function by moving its argument  $\mathbf{z}$ , which  $t$ , the parameters of the irregular singularities, move.

Kitaev and Korotkin [7] constructed a monodromy-preserving deformation of  $2 \times 2$  Fuchsian systems, whose deformation parameters are the positions of  $2g + 2$  regular singularities. Its  $\tau$ -function is expressed by the hyperelliptic  $\Theta$  function moving the period  $\mathbf{B}$ , which the positions of regular singularities move. When  $g = 1$ , the  $\tau$ -function is equivalent to Picard's solution of the sixth Painlevé equation by the Bäcklund transformation. The aim of this paper is to unify Miwa and Jimbo's result and Kitaev and Korotkin's result in the hyperelliptic case. We note that Deift, Its, Kapaev and Zhou [1] also constructed the isomonodromic deformations of  $2 \times 2$  Fuchsian systems in terms of the hyperelliptic  $\Theta$  functions.

This paper is organized as follows. In Section 1, we explain the basic properties of the  $\Theta$  function, the prime-form and the canonical bi-meromorphic differential.

In Section 2, we study an ordinary differential equation which is given by

$$\frac{d\Psi}{d\lambda} = \left( \sum_{j=1}^{2g+2} \frac{A_j}{\lambda - \lambda_j} - B_{-1} \right) \Psi(\lambda), \quad (0.2)$$

whose deformation parameters are  $\lambda_1, \lambda_2, \dots, \lambda_{2g+2}$  and diagonal elements of  $B_{-1}$ . Then, following Miwa, Jimbo and Ueno [5], we introduce the  $\tau$ -function.

In Section 3, we solve a class of Riemann-Hilbert (inverse monodromy) problem for special parameters. Furthermore, we prove that the solution  $\Psi(\lambda)$  has the following special monodromy data:

- (i) Stokes multipliers around  $\lambda = \infty$   $S_1^\infty = S_2^\infty = 1$ ,
- (ii) connection matrices around  $\lambda = \lambda_j$   $C_j$  ( $1 \leq j \leq 2g + 2$ ),
- (iii) the exponents of formal monodromy  $T_{\infty,0} = 0, T_{j,0} = \text{diag}(-\frac{1}{4}, \frac{1}{4})$  ( $1 \leq j \leq 2g + 2$ ).

In Section 4, we calculate the monodromy-preserving deformation which the solution  $\Psi(\lambda)$  satisfies and compute the coefficients,  $B_{-1}, A_j$  ( $1 \leq j \leq 2g + 2$ ).

In Section 5, we find the  $\tau$ -function. Section 5 consists of three subsections. Subsection 5.1 is devoted to  $H_t$ , the Hamiltonian at an irregular singular point  $\lambda = \infty$ . Subsection 5.2 is devoted to Fay's identities and Rauch's variational formulas. Subsection

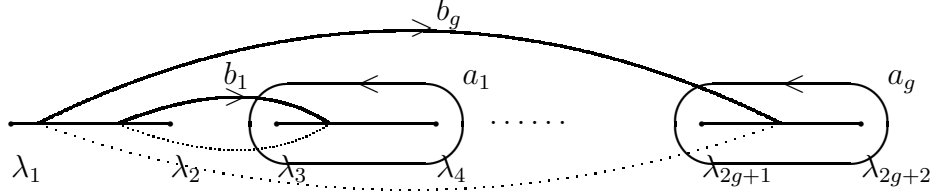


Figure 1: Branch cuts and canonical basis of cycles on the hyperelliptic curves,  $\mathcal{L}$ . Continuous and dashed paths lie on the first and second sheet of  $\mathcal{L}$ , respectively.

5.3 is devoted to  $H_j$  ( $1 \leq j \leq 2g+2$ ), the Hamiltonians on the deformation parameters  $\lambda_j$  ( $1 \leq j \leq 2g+2$ ). From  $H_t$  and  $H_j$  ( $1 \leq j \leq 2g+2$ ), we compute the  $\tau$ -function and prove Theorem 0.1.

In order to state our main theorem, we define hyperelliptic curves and the  $\Theta$  function and introduce the prime-form and the canonical bi-meromorphic differential, following Fay [2]. The hyperelliptic curves  $\mathcal{L}$  are

$$\omega^2 = \prod_{j=1}^{2g+2} (\lambda - \lambda_j),$$

whose branch points  $\lambda_i \in \mathbb{C}$  ( $1 \leq i \leq 2g+2$ ) are distinct and canonical homological basis  $\{a_j, b_j\}_{1 \leq j \leq g}$  are chosen according to Figure 1.

The basic holomorphic one forms are expressed by

$$dU_k^0 = \frac{\lambda^{k-1} d\lambda}{\omega}, \quad 1 \leq k \leq g.$$

Then, the  $g \times g$  matrices of a- and b-periods are given by

$$\mathcal{A}_{kj} = \oint_{a_j} dU_k^0, \quad \mathcal{B}_{kj} = \oint_{b_j} dU_k^0, \quad 1 \leq k, j \leq g.$$

Thus, the normalized holomorphic one forms are defined by

$$dU_k = \frac{1}{\omega} \sum_{j=1}^g (\mathcal{A}^{-1})_{kj} \lambda^{j-1} d\lambda \quad 1 \leq k \leq g,$$

which satisfy

$$\oint_{a_j} dU_k = \delta_{jk}.$$

Furthermore, from  $\mathcal{A}$  and  $\mathcal{B}$ , we can construct

$$\mathbf{B} = \mathcal{A}^{-1}\mathcal{B}.$$

Following Mumford [9], we define the  $\Theta$  function with characteristic  $[\mathbf{p}, \mathbf{q}]$  ( $\mathbf{p}, \mathbf{q} \in \mathbb{C}^g$ ) by

$$\Theta[\mathbf{p}, \mathbf{q}](\mathbf{z}|\mathbf{B}) = \sum_{\mathbf{m} \in \mathbb{Z}^g} \exp\{\pi i \langle \mathbf{B}(\mathbf{m} + \mathbf{p}), \mathbf{m} + \mathbf{p} \rangle + 2\pi i \langle \mathbf{z} + \mathbf{q}, \mathbf{m} + \mathbf{p} \rangle\}, \quad (0.3)$$

where  $\mathbf{z} \in \mathbb{C}^g$  and the sum extends over all integer vectors in  $\mathbb{C}^g$ .

Following Fay [2], we introduce the prime-form  $E(P, Q)$  and the canonical bi-meromorphic differential  $W(P, Q)$ . The prime-form is defined by

$$E(P, Q) = \frac{\Theta[\mathbf{p}^*, \mathbf{q}^*](U(P) - U(Q))}{h_*(P)h_*(Q)}$$

$$(h_*(P))^2 = \sum_{k=1}^g \frac{\partial \Theta[\mathbf{p}^*, \mathbf{q}^*]}{\partial z_k}(0|\mathbf{B}) dU_k(P) \quad \text{for } P, Q \in \mathcal{L},$$

where  $[\mathbf{p}^*, \mathbf{q}^*]$  is an arbitrary odd non-singular half integer characteristic. The canonical bi-meromorphic differential  $W(P, Q)$  is given by

$$W(P, Q) = dx_P dx_Q \log E(P, Q),$$

where  $dx_P, dx_Q$  are the exterior differentiations with respect to the local parameter of  $P, Q$ , respectively. When  $P = \infty^1$  and  $Q = \infty^2$ , we define the local coordinates of  $P, Q$  by

$$x_{\infty^1} = \frac{1}{\lambda}, \quad x_{\infty^2} = \frac{1}{\lambda},$$

and have

$$W(\infty^1, \infty^2) = \left( \frac{\partial^2}{\partial x_{\infty^1} \partial x_{\infty^2}} \log E(P, Q) \Big|_{P=\infty^1, Q=\infty^2} \right) dx_{\infty^1} dx_{\infty^2}.$$

According to Fay [2], the projective connection  $S(Q)$  is given by the equation

$$W(P, Q) = \left( \frac{1}{(x_P - x_Q)^2} + \frac{1}{6} S(Q) + O(x_P - x_Q) \right) dx_P dx_Q,$$

where  $P$  and  $Q$  have local coordinates  $x_P, x_Q$  in a neighborhood of  $Q \in \mathcal{L}$ . When  $Q = \infty^1$ , we define the local coordinate of  $Q$  by

$$x_P = x_Q = \frac{1}{\lambda} \quad \text{if } P \longrightarrow Q = \infty^1.$$

Our main theorem is as follows:

**Theorem 0.1.** *For the monodromy-preserving deformation (4.1), the  $\tau$ -function is*

$$\begin{aligned} \tau(\lambda_1, \dots, \lambda_{2g+2}; t) = & \Theta[\mathbf{p}, \mathbf{q}] (\mathbf{v}(t) | \mathbf{B}) (\det \mathcal{A})^{-\frac{1}{2}} \prod_{1 \leq j < k \leq 2g+2} (\lambda_j - \lambda_k)^{-\frac{1}{8}} \\ & \times \exp \left\{ \frac{t^2}{4} \left( \frac{1}{6} S(\infty^1) - \frac{W(\infty^1, \infty^2)}{dx_{\infty^1} dx_{\infty^2}} \right) \right\}, \end{aligned}$$

where

$$\mathbf{v}(t) = t \times \left( \frac{dU_1}{dx_{\infty^1}}(\infty^1), \frac{dU_2}{dx_{\infty^1}}(\infty^1), \dots, \frac{dU_g}{dx_{\infty^1}}(\infty^1) \right)$$

and  $x_{\infty^1} = \frac{1}{\lambda}$ , which is a local coordinate of  $\infty^1$ .

**Remark.** By setting  $t = 0$ , we obtain Kitaev and Korotkin's  $\tau$ -function in [7]. We had a branch point at  $\infty$  in [4], while we do not in this paper.

## 1 Hyperelliptic Curves and the $\Theta$ Function

In this section, we explain the detailed properties of the  $\Theta$  function, the prime-form  $E(P, Q)$  and the canonical bi-meromorphic differential  $W(P, Q)$ .

The  $\Theta$  function possesses the following periodicity properties:

$$\Theta[\mathbf{p}, \mathbf{q}](\mathbf{z} + \mathbf{e}_j | \mathbf{B}) = \exp\{2\pi i p_j\} \Theta[\mathbf{p}, \mathbf{q}](\mathbf{z} | \mathbf{B}) \quad (1.1)$$

$$\Theta[\mathbf{p}, \mathbf{q}](\mathbf{z} + \mathbf{B}\mathbf{e}_j | \mathbf{B}) = \exp\{-2\pi i q_j - \pi i \mathbf{B}_{jj}\} \Theta[\mathbf{p}, \mathbf{q}](\mathbf{z} | \mathbf{B}), \quad (1.2)$$

where

$$\mathbf{e}_j = {}^t(0, \dots, \overset{jth}{1}, \dots, 0).$$

We define the Abel map  $U(P) \in \mathbb{C}^g$   $P \in \mathcal{L}$  by

$$\begin{aligned} U(P) &= {}^t(U_1(P), U_2(P), \dots, U_g(P)) \\ U_j(P) &= \int_{\lambda_1}^P dU_j \quad (1 \leq j \leq g). \end{aligned}$$

For the canonical homological basis  $\{a_j, b_j\}_{1 \leq j \leq g}$  and the base point  $\lambda_1$ , the Riemann constants  $\mathbf{K} \in \mathbb{C}^g$  are as follows:

$$\mathbf{K} = \frac{1}{2} \mathbf{B} (\mathbf{e}_1 + \mathbf{e}_2 + \dots + \mathbf{e}_g) + \frac{1}{2} (\mathbf{e}_1 + 2\mathbf{e}_2 + \dots + g\mathbf{e}_g).$$

A characteristic  $[\mathbf{p}, \mathbf{q}]$  is a  $g \times 2$  matrix of complex numbers which is given by

$$[\mathbf{p}, \mathbf{q}] = \begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \\ \vdots & \vdots \\ p_g & q_g \end{bmatrix},$$

where

$$\mathbf{p} = {}^t(p_1, p_2, \dots, p_g), \mathbf{q} = {}^t(q_1, q_2, \dots, q_g).$$

We consider  $p_i, q_i$  ( $1 \leq i \leq g$ ) as elements of  $\mathbb{C}^g/\mathbb{Z}^g$ . If all the components are half-integers,  $[\mathbf{p}, \mathbf{q}]$  is called a half-integer characteristic. A half-integer characteristic is in one-to-one correspondence with a half-period  $\mathbf{Bp} + \mathbf{q}$ . If the scalar product  $4\langle \mathbf{p}, \mathbf{q} \rangle$  is odd, then the characteristic is called odd and the related  $\Theta$  function is odd with respect to its argument  $\mathbf{z}$ . If this scalar product is even, then the characteristic is called even and the related  $\Theta$  function is even with respect to its argument  $\mathbf{z}$ .

The odd characteristics which are important for us in the sequel correspond to any subset

$$S = \{\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_{g-1}}\},$$

whose components are arbitrary distinct branch points. The odd half-period corresponding to the subset  $S$  is expressed by

$$\mathbf{Bp}^S + \mathbf{q}^S = U(\lambda_{i_1}) + U(\lambda_{i_2}) + \dots + U(\lambda_{i_{g-1}}) - \mathbf{K}.$$

Analogously, we shall be interested in the even half-period corresponding to the subset

$$T = \{\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_{g+1}}\},$$

which consists of arbitrary  $g + 1$  branch points. The even half-period is given by

$$\mathbf{Bp}^T + \mathbf{q}^T = U(\lambda_{i_1}) + U(\lambda_{i_2}) + \dots + U(\lambda_{i_{g+1}}) - \mathbf{K}.$$

We fix the choice of  $T$  and define

$$\{\lambda_{j_1}, \lambda_{j_2}, \dots, \lambda_{j_{g+1}}\} := \{\lambda_1, \lambda_2, \dots, \lambda_{2g+2}\} \setminus T.$$

We explain the detailed property of  $E(P, Q)$ . On page 13–14 of Fay [2], we find

$$\begin{aligned} E(P, Q) &= \frac{\Theta[\mathbf{p}^T, \mathbf{q}^T](U(P) - U(Q))}{\Theta[\mathbf{p}^T, \mathbf{q}^T](\mathbf{0})m_T(P, Q)}, \\ m_T(P, Q) &= \frac{\omega(Q) \prod_{k=1}^{g+1} (\lambda(P) - \lambda_{i_k}) + \omega(P) \prod_{k=1}^{g+1} (\lambda(Q) - \lambda_{i_k})}{2(\lambda(Q) - \lambda(P))} \\ &\quad \times \left[ \frac{d\lambda(P)d\lambda(Q)}{\omega(P)\omega(Q) \prod_{k=1}^{g+1} (\lambda(P) - \lambda_{i_k})(\lambda(Q) - \lambda_{i_k})} \right]^{\frac{1}{2}}, \end{aligned}$$

because  $\mathcal{L}$  is hyperelliptic.

## 2 The Schlesinger System

We study an ordinary differential equation which is given by

$$\frac{d\Psi}{d\lambda} = \left( \sum_{j=1}^{2g+2} \frac{A_j}{\lambda - \lambda_j} - B_{-1} \right) \Psi(\lambda), \quad (2.1)$$

where  $A_1, A_2, \dots, A_{2g+2}, B_{-1} \in sl(2, \mathbb{C})$  are independent of  $\lambda$ .

The monodromy data of (2.1) are as follows:

- (i) Stokes multipliers around  $\lambda = \infty$   $S_1^\infty = S_2^\infty$ ;
- (ii) connection matrices around  $\lambda_j$   $C_j$ , ( $j = 1, 2, \dots, 2g+2$ );
- (iii) the exponents of formal monodromy  $T_{\infty,0}, T_{j,0}$  ( $j = 1, 2, \dots, 2g+2$ ).

In the next section, we obtain a convergent series around  $\lambda = \infty, \lambda_j$  ( $j = 1, 2, \dots, 2g+2$ ) which are expressed by

$$\Psi(\lambda) = (1 + O(\frac{1}{\lambda})) \exp T^\infty(\lambda) = \hat{\Psi}^\infty(\lambda) \exp T^\infty(\lambda), \quad (2.2)$$

$$\begin{aligned} \Psi(\lambda) &= G_j(1 + O(\lambda - \lambda_j)) \exp T_j(\lambda) = G_j \hat{\Psi}_j(\lambda) \exp T_{j,0}(\lambda), \\ &\quad (j = 1, 2, \dots, 2g+2), \end{aligned} \quad (2.3)$$

where

$$T^\infty(\lambda) = \begin{pmatrix} -\frac{t}{2} & \\ & \frac{t}{2} \end{pmatrix} \lambda + T_{\infty,0} \log(\frac{1}{\lambda}), \quad (2.4)$$

$$T_j(\lambda) = T_{j,0} \log(\lambda - \lambda_j) \quad (j = 1, 2, \dots, 2g+2). \quad (2.5)$$

For the deformation parameters  $t, \lambda_1, \lambda_2, \dots, \lambda_{2g+2}$ , the closed one form is defined by

$$\Omega = \omega_\infty + \omega_{\lambda_1} + \omega_{\lambda_2} + \dots + \omega_{\lambda_{2g+2}} \quad (2.6)$$

$$= H_t dt + H_1 d\lambda_1 + H_2 d\lambda_2 + \dots + H_{2g+2} d\lambda_{2g+2}, \quad (2.7)$$

where

$$\omega_\infty = -\text{Res}_{\lambda=\infty} \text{tr} \hat{\Psi}^\infty(\lambda)^{-1} \frac{\partial \hat{\Psi}^\infty}{\partial \lambda}(\lambda) dT^\infty(\lambda), \quad (2.8)$$

$$\omega_{\lambda_j} = -\text{Res}_{\lambda=\lambda_j} \text{tr} \hat{\Psi}_j(\lambda)^{-1} \frac{\partial \hat{\Psi}_j}{\partial \lambda}(\lambda) dT_j(\lambda) \quad (j = 1, 2, \dots, 2g+2), \quad (2.9)$$

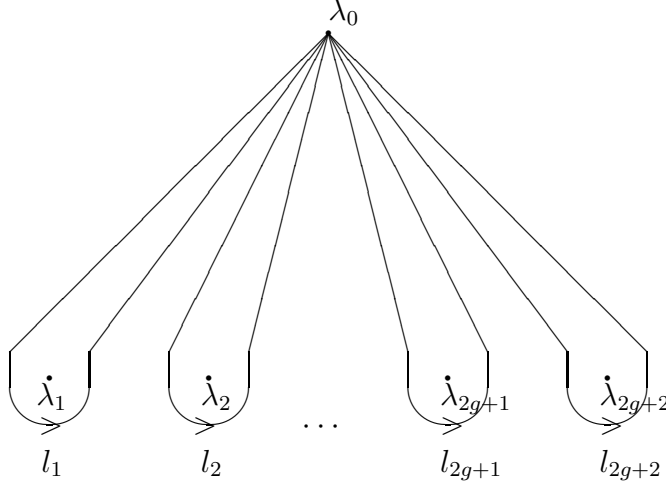


Figure 2: Generators of  $\pi_1(\mathbb{CP}^1 \setminus \{\lambda_1, \dots, \lambda_{2g+2}\}, \lambda_0)$

and  $d$  is the exterior differentiation with respect to the deformation parameters  $t, \lambda_1, \lambda_2, \dots, \lambda_{2g+2}$ . Especially, we can write

$$\omega_{\lambda_j} = \left[ \text{Res}_{\lambda=\lambda_j} \frac{1}{2} \text{tr} \left( \frac{d\Psi}{d\lambda} \Psi^{-1} \right)^2 \right] d\lambda_j. \quad (2.10)$$

Then, from the closed one form  $\Omega$ , the  $\tau$ -function is defined by

$$\Omega := d \log \tau(\lambda_1, \lambda_2, \dots, \lambda_{2g+2}, t). \quad (2.11)$$

### 3 The Riemann-Hilbert Problem for Special Parameters

In this section, we concretely construct a  $2 \times 2$  matrix valued function  $\Psi(\lambda)$ , whose monodromy data, (i) Stokes multipliers, (ii) connection matrices, (iii) exponents of formal monodromy, are independent of the deformation parameters  $t, \lambda_1, \lambda_2, \dots, \lambda_{2g+2}$ .

The involution of  $\mathcal{L}$  is defined by

$$* : (\lambda, \omega) \longrightarrow (\lambda, -\omega).$$

Then, from the  $\Theta$  function and  $W(P, Q)$ , we define the  $2 \times 2$  matrix valued function  $\Phi(P)$  by

$$\Phi(P) = \begin{pmatrix} \varphi(P) & \varphi(P^*) \\ \psi(P) & \psi(P^*) \end{pmatrix}, \quad (3.1)$$



where

$$\begin{aligned}
\varphi(P) &= \Theta[\mathbf{p}, \mathbf{q}](U(P) + U(P_\varphi) + \mathbf{v}(t))\Theta[\mathbf{p}^S, \mathbf{q}^S](U(P) - U(P_\varphi)) \\
&\quad \times \exp -\frac{t}{2}\left\{-\int_{\lambda_1}^P W(P, \infty^1) + \int_{\lambda_1}^P W(P, \infty^2)\right\}, \\
\psi(P) &= \Theta[\mathbf{p}, \mathbf{q}](U(P) + U(P_\psi) + \mathbf{v}(t))\Theta[\mathbf{p}^S, \mathbf{q}^S](U(P) - U(P_\psi)) \\
&\quad \times \exp -\frac{t}{2}\left\{-\int_{\lambda_1}^P W(P, \infty^1) + \int_{\lambda_1}^P W(P, \infty^2)\right\}, \\
\mathbf{v}(t) &= t \times \left(\frac{dU_1}{dx_{\infty^1}}(\infty^1), \frac{dU_2}{dx_{\infty^1}}(\infty^1), \dots, \frac{dU_g}{dx_{\infty^1}}(\infty^1)\right),
\end{aligned}$$

where  $P_\varphi, P_\psi$  are arbitrary points of  $\mathcal{L}$  and  $x_{\infty^1} = \frac{1}{\lambda}$ , which is a local coordinate of  $\infty^1$ .

**Proposition 3.1.** (1) The function  $\Phi(P)$  is invertible outside of the branch points  $\lambda_1, \lambda_2, \dots, \lambda_{2g+2}$ .

(2)  $\det \Phi(P)$  has zeros at  $\lambda_j \notin S$  with the first order and has zeros at  $\lambda_j \in S$  with the third order.

(3) The function  $\Phi(P)$  transforms as follows with respect to the tracing along the canonical homological basis,  $a_j, b_j$  ( $j = 1, 2, \dots, g$ ):

$$T_{a_j}[\Phi(P)] = \Phi(P) \begin{pmatrix} \exp\{2\pi i(p_j + p_j^S)\} & \\ & \exp\{-2\pi i(p_j + p_j^S)\} \end{pmatrix}, \quad (3.2)$$

$$T_{b_j}[\Phi(P)] = \Phi(P) \begin{pmatrix} \exp\{-2\pi i(q_j + q_j^S)\} & \\ & \exp\{2\pi i(q_j + q_j^S)\} \end{pmatrix} \exp\{-2\pi i\mathbf{B}_{jj} - 4\pi iU_j(P)\}, \quad (3.3)$$

where  $T_l$  denotes the operator of analytic continuation along the contour  $l$ .

*Proof.* By using the periodicity (1.1) and (1.2), we obtain

$$\begin{aligned}
T_{a_j}[\varphi(P)] &= \exp\{2\pi i(p_j + p_j^S)\}\varphi(P), \\
T_{b_j}[\varphi(P)] &= \exp\{-2\pi i(q_j + q_j^S) - 2\pi i\mathbf{B}_{jj} - 4\pi iU_j(P)\}\varphi(P).
\end{aligned}$$

We deduce the same transformation laws for  $\psi(P)$ .

The actions of the involution  $*$  on  $\{a_j, b_j\}$  and  $dU_j$  are given by

$$a_j^* = -a_j, \quad b_j^* = -b_j, \quad dU_j(P^*) = -dU_j(P), \quad (j = 1, 2, \dots, g),$$

respectively. Therefore, we get

$$T_{a_j}[\varphi(P^*)] = \exp\{-2\pi i(p_j + p_j^S)\}\varphi(P^*), \quad (3.4)$$

$$T_{b_j}[\varphi(P^*)] = \exp\{2\pi i(q_j + q_j^S) - 2\pi i\mathbf{B}_{jj} - 4\pi iU_j(P)\}\varphi(P^*). \quad (3.5)$$

We deduce the same transformation laws for  $\psi(P^*)$ . Then, we complete the proof of (3).

The equations (3.2) and (3.3) imply that

$$\begin{aligned} T_{a_j}[\det \Phi(P)] &= \det \Phi(P), \\ T_{b_j}[\det \Phi(P)] &= \det \Phi(P) \exp\{-4\pi i\mathbf{B}_{jj} - 8\pi iU_j(P)\}, \end{aligned}$$

which imply that

$$\frac{1}{2\pi i} \oint_{\partial\mathcal{L}} \frac{d(\det \Phi(P))}{\det \Phi(P)} = 4g.$$

Thus, it follows that

$$3(g-1) + g + 3 = 4g,$$

because  $\det \Phi(P)$  has zeros at the branch points  $\lambda_j$  and has zeros at  $\lambda_j \in S$  of the order of at least three. Therefore, it follows that  $\det \Phi(P)$  does not vanish outside of the branch points and that  $\det \Phi(P)$  has zeros at  $\lambda_j \notin S$  with the first order and has zeros at  $\lambda_j \in S$  with the third order, which complete the proof of (1) and (2), respectively.  $\square$

In order to normalize  $\Phi(\lambda)$  near  $\lambda = \infty$ , we have

**Lemma 3.2.** *For  $P, Q \in \mathcal{L}$ ,*

$$W(P, Q) = \left( \frac{1}{(x_P - x_Q)^2} + \frac{1}{6}S(Q) + O(x_P - x_Q) \right) dx_P dx_Q, \quad (3.6)$$

$$\begin{aligned} \frac{1}{6}S(Q) &= \frac{1}{6}\{\lambda, x_Q\}(Q) + \frac{1}{16} \left( \frac{d}{dx_Q} \log \prod_{k=1}^{g+1} \frac{\lambda - \lambda_{i_k}}{\lambda - \lambda_{j_k}}(Q) \right)^2 \\ &\quad - \sum_{k,l=1}^g \frac{\partial^2}{\partial z_k \partial z_l} \log \Theta[\mathbf{p}^T, \mathbf{q}^T](0) \frac{dU_k}{dx_Q}(Q) \frac{dU_l}{dx_Q}(Q), \end{aligned} \quad (3.7)$$

where  $x_P, x_Q$  are local coordinates of  $P, Q \in \mathcal{L}$  and

$$\{\lambda, x\} = \frac{\lambda'''}{\lambda'} - \frac{3}{2} \left( \frac{\lambda''}{\lambda'} \right)^2.$$

*Proof.* See pp. 20 in [2].  $\square$

In Lemma 3.2. we set

$$x_P = x_Q = \frac{1}{\lambda}, \quad Q = \infty^1,$$

and take the limit  $P \longrightarrow \infty^1$ . Furthermore, we define the constant terms  $c_{\infty^1}, c_{\infty^2}$  by

$$\begin{aligned} \int_{\lambda_1}^P W(P, \infty^1) &= -\lambda + c_{\infty^1} + \frac{1}{6} S(\infty^1) \lambda^{-1} + \dots \quad \text{near } \lambda = \infty^1, \\ \int_{\lambda_1}^P W(P, \infty^2) &= c_{\infty^2} + \frac{W(\infty^1, \infty^2)}{dx_{\infty^1} dx_{\infty^2}} \lambda^{-1} + \dots \quad \text{near } \lambda = \infty^1. \end{aligned}$$

Therefore,  $\Phi(\lambda)$  can be developed near  $\lambda = \infty$  as

$$\Phi(\lambda) = \left( G^\infty + O\left(\frac{1}{\lambda}\right) \right) \exp \begin{pmatrix} -\frac{t}{2}\lambda & \\ & \frac{t}{2}\lambda \end{pmatrix},$$

where  $G^\infty$  is a  $2 \times 2$  matrix whose matrix elements are given by

$$\begin{aligned} (G^\infty)_{11} &= \Theta[\mathbf{p}, \mathbf{q}](U(\infty^1) + U(P_\varphi) + \mathbf{v}(t)) \Theta[\mathbf{p}^S, \mathbf{q}^S](U(\infty^1) - U(P_\varphi)) \exp \left\{ \frac{t}{2} (c_{\infty^1} - c_{\infty^2}) \right\}, \\ (G^\infty)_{21} &= \Theta[\mathbf{p}, \mathbf{q}](U(\infty^1) + U(P_\psi) + \mathbf{v}(t)) \Theta[\mathbf{p}^S, \mathbf{q}^S](U(\infty^1) - U(P_\psi)) \exp \left\{ \frac{t}{2} (c_{\infty^1} - c_{\infty^2}) \right\}, \\ (G^\infty)_{12} &= \Theta[\mathbf{p}, \mathbf{q}](U(\infty^2) + U(P_\varphi) + \mathbf{v}(t)) \Theta[\mathbf{p}^S, \mathbf{q}^S](U(\infty^2) - U(P_\varphi)) \exp \left\{ -\frac{t}{2} (c_{\infty^1} - c_{\infty^2}) \right\}, \\ (G^\infty)_{22} &= \Theta[\mathbf{p}, \mathbf{q}](U(\infty^2) + U(P_\psi) + \mathbf{v}(t)) \Theta[\mathbf{p}^S, \mathbf{q}^S](U(\infty^2) - U(P_\psi)) \exp \left\{ -\frac{t}{2} (c_{\infty^1} - c_{\infty^2}) \right\}. \end{aligned}$$

Proposition 3.1 shows that

$$\det G^\infty = \det \Phi(\infty) \neq 0.$$

We define a matrix valued function  $\Psi(\lambda)$  by

$$\Psi(P) = \frac{\sqrt{\det \Phi(\infty)}}{\sqrt{\det \Phi(P)}} (G^\infty)^{-1} \Phi(P). \quad (3.8)$$

The expansions of  $\Psi(\lambda)$  near  $\lambda = \infty$  are as follows:

**Lemma 3.3.**

$$\begin{aligned} \Psi(\lambda) &= \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O\left(\frac{1}{\lambda}\right) \right) \exp T^\infty(\lambda), \\ T^\infty(\lambda) &= T_{-1}^\infty \lambda, \quad T_{-1}^\infty = \begin{pmatrix} -\frac{t}{2} & \\ & \frac{t}{2} \end{pmatrix}, \end{aligned}$$

where the Taylor series of  $\Psi(\lambda) \exp \{-T^\infty(\lambda)\}$  is convergent. Especially, if  $P_\varphi = \infty^1$  and  $P_\psi = \infty^2$ ,

$$\begin{aligned}\Psi(\lambda) &= \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \Psi_{-1}^\infty \lambda^{-1} + \dots \right) \exp T^\infty(\lambda), \\ (\Psi_{-1}^\infty)_{11} &= \sum_{k=1}^g \frac{\partial}{\partial z_k} \log \Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t)) \frac{dU_k}{dx_{\infty^1}}(\infty^1) + \frac{t}{2} \left( \frac{1}{6} S(\infty^1) - \frac{W(\infty^1, \infty^2)}{dx_{\infty^1} dx_{\infty^2}} \right), \\ (\Psi_{-1}^\infty)_{21} &= \frac{i}{E(\infty^2, \infty^1)} \frac{\Theta[\mathbf{p}, \mathbf{q}](2U(\infty^1) + \mathbf{v}(t))}{\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))} \exp \{t(c_{\infty^1} - c_{\infty^2})\}, \\ (\Psi_{-1}^\infty)_{12} &= \frac{i}{E(\infty^1, \infty^2)} \frac{\Theta[\mathbf{p}, \mathbf{q}](2U(\infty^2) + \mathbf{v}(t))}{\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))} \exp \{-t(c_{\infty^1} - c_{\infty^2})\}, \\ (\Psi_{-1}^\infty)_{22} &= \sum_{k=1}^g \frac{\partial}{\partial z_k} \log \Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t)) \frac{dU_k}{dx_{\infty^2}}(\infty^2) - \frac{t}{2} \left( \frac{1}{6} S(\infty^1) - \frac{W(\infty^1, \infty^2)}{dx_{\infty^1} dx_{\infty^2}} \right).\end{aligned}$$

In the following theorem, we determine the monodromy matrices and the Stokes matrices of  $\Psi(\lambda)$ .

**Theorem 3.4.** *For  $\nu = 1, 2, \dots, 2g+2$ , the monodromy matrix  $M_\nu$  of  $\Psi(\lambda)$  corresponding to the contour  $l_\nu$  is given by*

$$M_\nu = \begin{pmatrix} 0 & m_\nu \\ -m_\nu^{-1} & 0 \end{pmatrix}, \quad (3.9)$$

where

$$\begin{aligned}m_1 &= -i, \quad m_2 = i(-1)^{g+1} \exp \left\{ -2\pi i \sum_{k=1}^g p_k \right\}, \\ m_{2j+1} &= i(-1)^g \exp \left\{ 2\pi i q_j - 2\pi i \sum_{k=j}^g p_k \right\}, \\ m_{2j+2} &= i(-1)^{g+1} \exp \left\{ 2\pi i q_j - 2\pi i \sum_{k=j+1}^g p_k \right\},\end{aligned}$$

for  $j = 1, 2, \dots, g$ . The Stokes matrices are expressed by

$$S_1^\infty = S_2^\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

*Proof.* By the involution  $*$ , we get

$$\Psi(\lambda)M_1 = \Psi(\lambda) \begin{pmatrix} & i \\ i & \end{pmatrix}, \text{ or } = \Psi(\lambda) \begin{pmatrix} & -i \\ -i & \end{pmatrix}.$$

We define

$$M_1 = \begin{pmatrix} & m_1 \\ -m_1^{-1} & \end{pmatrix} = \begin{pmatrix} & -i \\ -i & \end{pmatrix}.$$

Proposition 3.1 implies that

$$\begin{aligned} T_{a_j} [\Psi(\lambda)] &= \Psi(\lambda) M_{2j+2} M_{2j+1} \\ &= \Psi(\lambda) \frac{T_{l_{2j+1} o l_{2j+2}} [\sqrt{\det \Phi(P)}]}{\sqrt{\det \Phi(P)}} \exp\{2\pi i(p_j + p_j^S)\} \sigma_3, \end{aligned}$$

and

$$\begin{aligned} T_{-b_j+b_{j-1}} [\Psi(\lambda)] &= \Psi(\lambda) M_{2j+1} M_{2j} \\ &= \Psi(\lambda) \frac{T_{l_{2j} o l_{2j+1}} [\sqrt{\det \Phi(P)}]}{\sqrt{\det \Phi(P)}} \exp\{2\pi i(q_j - q_{j-1} + q_j - q_{j-1})\} \sigma_3, \end{aligned}$$

where  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . In order to determine the monodromy matrices, we have

$$\begin{aligned} U(\lambda_1) &= 0, & U(\lambda_2) &= \frac{1}{2} \sum_{k=1}^g \mathbf{e}_k, \\ U(\lambda_{2j+1}) &= \frac{1}{2} \mathbf{B} \mathbf{e}_j + \frac{1}{2} \sum_{k=j}^g \mathbf{e}_k, & U(\lambda_{2j+2}) &= \frac{1}{2} \mathbf{B} \mathbf{e}_j + \frac{1}{2} \sum_{k=j+1}^g \mathbf{e}_k, \quad (j = 1, 2, \dots, g). \end{aligned}$$

Then, we get

$$p_j^S = \frac{1}{2} (\delta_{2j+1} + \delta_{2j+2} + 1), \quad q_{j+1} - q_j = \frac{1}{2} (\delta_{2j+2} + \delta_{2j+3} + 1),$$

where

$$\begin{cases} \delta_j = 1 & \text{if } \lambda_j \in S \\ \delta_j = 0 & \text{if } \lambda_j \notin S. \end{cases}$$

The function  $\sqrt{\det \Phi(P)}$  transforms with respect to the tracing along the cycles  $l_j$  in the following way:

$$\begin{cases} T_{l_{2j+1} o l_{2j+2}} [\sqrt{\det \Phi(P)}] = \exp\{\pi i(\delta_{2j+1} + \delta_{2j+2} + 1)\} \sqrt{\det \Phi(P)}, \\ T_{l_{2j} o l_{2j+1}} [\sqrt{\det \Phi(P)}] = \exp\{\pi i(\delta_{2j+2} + \delta_{2j+3} + 1)\} \sqrt{\det \Phi(P)}. \end{cases} \quad (3.10)$$

In order to prove (3.10), we have only to note that if  $\lambda_j$  is in  $S$ , from Proposition 3.1,  $\det \Phi(P)$  has a zero of order one at  $\lambda = \lambda_j$  and that if  $\lambda_j$  is not in  $S$ , from Proposition 3.1,  $\det \Phi(P)$  has a zero of order three at  $\lambda = \lambda_j$ . Therefore, we obtain

$$\begin{cases} M_{2j+2}M_{2j+1} = \exp\{2\pi i\sigma_3\} \\ M_{2j+1}M_{2j} = \exp\{2\pi i(q_j - q_{j-1})\sigma_3\}. \end{cases}$$

By considering

$$M_{2g+2}M_{2g+1} \cdots M_1 = I,$$

we get the monodromy matrices.

Because  $\Psi(\lambda) \exp\{-T^\infty(\lambda)\}$  can be developed near  $\lambda = \infty$  as a convergent series, the Stokes multipliers are  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .  $\square$

We can describe the monodromy data of  $\Psi(\lambda)$  in the following way.

**Corollary 3.5.**  *$\Psi(\lambda)$  has the following monodromy data:*

- (i) *Stokes multipliers*  $S_1^\infty = S_2^\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,
- (ii) *connection matrices*  $C_j = \frac{1}{\sqrt{2im_j}} \begin{pmatrix} 1 & im_j \\ -1 & im_j \end{pmatrix}$  ( $j = 1, 2, \dots, 2g+2$ ),
- (iii) *exponents of formal monodromy*  $T_{j,0} = \text{diag}(-\frac{1}{4}, \frac{1}{4})$  ( $j = 1, 2, \dots, 2g+2$ ).

*Especially, the developments of  $\Psi(\lambda)$  near  $\lambda = \lambda_j$  ( $j = 1, 2, \dots, 2g+2$ ) are expressed by*

$$\begin{aligned} \Psi(\lambda) &= G_j(1 + O(\lambda - \lambda_j)) \exp T_j(\lambda) C_j, \\ T_j(\lambda) &= T_{j,0} \log(\lambda - \lambda_j). \end{aligned}$$

*Proof.* (i) is clear. (ii) and (iii) can be obtained by diagonalizing the monodromy matrices  $M_j$  ( $j = 1, 2, \dots, 2g+2$ ).  $\square$

This corollary means that the monodromy data of  $\Psi(\lambda)$  are independent of the deformation parameters  $t, \lambda_1, \lambda_2, \dots, \lambda_{2g+2}$ .

## 4 Monodromy-Preserving Deformation

In this section, we prove that  $\Psi(\lambda)$  satisfies an ordinary differential equation which is expressed by

$$\frac{d\Psi}{d\lambda} = \left( \sum_{j=1}^{2g+2} \frac{A_j}{\lambda - \lambda_j} - B_{-1} \right) \Psi(\lambda),$$

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and concretely determine the coefficients  $B_{-1}, A_j$  ( $1 \leq j \leq 2g + 2$ ).

We note that the monodromy matrices and the Stokes matrices are independent of the parameters  $P_\varphi, P_\phi, p_k^S, q_k^S$  ( $1 \leq k \leq g$ ) in Theorem 3.4. Thus, in this section, we set  $P_\varphi = \infty^2, P_\phi = \infty^1$  and take  $S_j$  so that  $\lambda_j$  is not in  $S_j$  because of the uniqueness of a solution of the Riemann-Hilbert problem. Therefore, we get

$$\Psi(\lambda) = \frac{1}{\sqrt{\det \Phi^\infty(\lambda)}} \Phi^\infty(\lambda),$$

where

$$\begin{aligned} \Psi(P) &= \begin{pmatrix} \varphi_j^\infty(P) & \varphi_j^\infty(P^*) \\ \psi_j^\infty(P) & \psi_j^\infty(P^*) \end{pmatrix}, \\ \varphi_j^\infty(P) &= \frac{\Theta[\mathbf{p}, \mathbf{q}](U(P) + U(\infty^2) + \mathbf{v}(t)) \Theta[\mathbf{p}^{S_j}, \mathbf{q}^{S_j}](U(P) - U(\infty^2))}{\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t)) \Theta[\mathbf{p}^S, \mathbf{q}^S](-2U(\infty^2))} \\ &\quad \times \exp\left\{-\frac{t}{2}(c_{\infty^1} - c_{\infty^2})\right\} \exp \Pi(P), \\ \psi_j^\infty(P) &= \frac{\Theta[\mathbf{p}, \mathbf{q}](U(P) + U(\infty^1) + \mathbf{v}(t)) \Theta[\mathbf{p}^{S_j}, \mathbf{q}^{S_j}](U(P) - U(\infty^1))}{\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t)) \Theta[\mathbf{p}^S, \mathbf{q}^S](-2U(\infty^1))} \\ &\quad \times \exp\left\{\frac{t}{2}(c_{\infty^1} - c_{\infty^2})\right\} \exp \Pi(P). \end{aligned}$$

**Theorem 4.1.**  $\Psi(\lambda)$  satisfies the following ordinary differential equation:

$$\frac{d\Psi}{d\lambda} = \left( \sum_{j=1}^{2g+2} \frac{A_j}{\lambda - \lambda_j} - B_{-1} \right) \Psi(\lambda), \quad (4.1)$$

where

$$\begin{aligned} B_{-1} &= \text{diag}\left(\frac{t}{2}, -\frac{t}{2}\right), \\ A_j &= -\frac{1}{4} F_j^\infty \sigma_3 (F_j^\infty)^{-1}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ F_j^\infty &= \begin{pmatrix} \varphi_j^\infty(\lambda_j) & \frac{d}{dx_j} \varphi_j^\infty(\lambda_j) \\ \psi_j^\infty(\lambda_j) & \frac{d}{dx_j} \psi_j^\infty(\lambda_j) \end{pmatrix}, \end{aligned}$$

and  $x_j = \sqrt{\lambda - \lambda_j}$ .

*Proof.* From Lemma 3.3, it follows that

$$\Psi_\lambda \Psi^{-1} = \begin{pmatrix} -\frac{t}{2} & \\ & \frac{t}{2} \end{pmatrix} + O(\lambda^{-1}) := -B_{-1} + O(\lambda^{-1}).$$

Near  $\lambda = \lambda_j$ , we have

$$\begin{cases} \varphi_j^\infty(P) = \varphi_j(\lambda_j) + \sqrt{\lambda - \lambda_j} \frac{d}{dx_j} \varphi_j(\lambda_j) + \cdots \\ \psi_j^\infty(P) = \psi_j(\lambda_j) + \sqrt{\lambda - \lambda_j} \frac{d}{dx_j} \psi_j(\lambda_j) + \cdots, \end{cases}$$

which implies that

$$\det \Phi^\infty(P) = -2\sqrt{\lambda - \lambda_j} \det F_j^\infty + O(\lambda - \lambda_j).$$

From the definition of  $S_j$ , it follows that  $\det F_j^\infty \neq 0$ .

We set

$$\Psi(\lambda) := G_j(1 + O(\lambda - \lambda_j))(\lambda - \lambda_j) \begin{pmatrix} -\frac{1}{4} & \\ & \frac{1}{4} \end{pmatrix} C_j.$$

Then, we get

$$\begin{aligned} G_j &= \Psi(\lambda) C_j^{-1} (\lambda - \lambda_j) \begin{pmatrix} \frac{1}{4} & \\ & -\frac{1}{4} \end{pmatrix} \Bigg|_{\lambda=\lambda_j} = (-2 \det F_j^\infty)^{-\frac{1}{2}} (1 + O(\sqrt{\lambda - \lambda_j})) \\ &\quad \times \Phi^\infty(P) C_j^{-1} (\lambda - \lambda_j) \begin{pmatrix} 1 & \\ & -\frac{1}{2} \end{pmatrix} \Bigg|_{\lambda=\lambda_j} \\ &= (-2 \det F_j^\infty)^{-\frac{1}{2}} \times 2im_j F_j^\infty. \end{aligned}$$

Thus, we obtain

$$\Psi_\lambda \Psi^{-1} = -\frac{1}{4} \frac{1}{\lambda - \lambda_j} F_j^\infty \sigma_3 (F_j^\infty)^{-1} + \cdots = \frac{A_j}{\lambda - \lambda_j} + \cdots.$$

□

From Theorem 4.1, we can obtain the following deformation equation:

**Corollary 4.2.** *The deformation equation of the monodromy-preserving deformation (4.1) is as follows. For  $j, k = 1, 2, \dots, 2g + 2$ ,*

$$\begin{aligned} dA_j &= [\Theta_j, A_j], \\ dF_j^\infty &= \Theta_j A_j, \\ \Theta_j &= \sum_{k \neq j} A_k \frac{d\lambda_k - d\lambda_j}{\lambda_k - \lambda_j} - [\Psi_{-1}^\infty, dB_{-1}] - d(\lambda_j B_{-1}), \end{aligned}$$

where “ $d$ ” is the exterior differentiation with respect to the deformation parameters,  $t, \lambda_1, \lambda_2, \dots, \lambda_{2g+2}$ .

*Proof.* See [5].

□



## 5 The $\tau$ -Function for the Schlesinger System

In this section, we calculate the  $\tau$ -function for the monodromy-preserving deformation (4.1). This section consists of three subsections. Subsection 5.1 is devoted to the Hamiltonian  $H_t$ . Subsection 5.2, 5.3 is devoted to the Hamiltonian  $H_j$  ( $j = 1, 2, \dots, 2g + 2$ ). In subsection 5.2, we quote Fay's identities and Rauch's variational formulas in order to compute the  $\tau$ -function. In subsection 5.3, we calculate  $H_j$  and the  $\tau$ -function.

### 5.1 The Hamiltonian at the Irregular Singular Point

In this subsection, we prove Proposition 5.1, where we compute  $\omega_\infty$  and the Hamiltonian  $H_t$ .

**Proposition 5.1.**

$$\begin{aligned} \omega_\infty &= \left( \frac{1}{\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))} \sum_{k=1}^g \frac{\partial}{\partial z_k} \{ \Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t)) \} \frac{dU_k}{dx_{\infty^1}}(\infty^1) + \frac{t}{2} \left( \frac{1}{6} S(\infty^1) - \frac{W(\infty^1, \infty^2)}{dx_{\infty^1} dx_{\infty^2}} \right) \right) dt \\ &= H_t dt. \end{aligned}$$

*Proof.* We define

$$\Pi(P) = -\frac{t}{2} \left\{ - \int_{\lambda_1}^P W(P, \infty^1) + \int_{\lambda_1}^P W(P, \infty^2) \right\}, \quad (5.1)$$

and let  $\hat{\Pi}(P)$  denote the regular part of  $\Pi(P)$  around  $\lambda = \infty^1$  which is given by

$$\hat{\Pi}(P) = -\frac{t}{2} \left\{ \text{const} + \left( -\frac{1}{6} S(\infty^1) + \frac{W(\infty^1, \infty^2)}{dx_{\infty^1} dx_{\infty^2}} \right) \lambda^{-1} + \dots \right\}. \quad (5.2)$$

Furthermore, we set

$$\begin{aligned} \hat{\varphi}(P) &= \Theta[\mathbf{p}, \mathbf{q}](U(P) + U(P_\varphi) + \mathbf{v}(t)) \Theta[\mathbf{p}^S, \mathbf{q}^S](U(P) - U(P_\varphi)), \\ \hat{\psi}(P) &= \Theta[\mathbf{p}, \mathbf{q}](U(P) + U(P_\psi) + \mathbf{v}(t)) \Theta[\mathbf{p}^S, \mathbf{q}^S](U(P) - U(P_\psi)). \end{aligned}$$

Then, we get

$$\begin{aligned} \Psi(P) &= \frac{\sqrt{\det \Phi(\infty)}}{\sqrt{\det \Phi(P)}} (G^\infty)^{-1} \Phi(P) \\ &= \frac{\sqrt{\det \Phi(\infty)}}{\sqrt{\det \Phi(P)}} (G^\infty)^{-1} \begin{pmatrix} \hat{\varphi}(P) \exp(\hat{\Pi}(P)) & \hat{\varphi}(P^*) \exp(\hat{\Pi}(P^*)) \\ \hat{\psi}(P) \exp(\hat{\Pi}(P)) & \hat{\psi}(P^*) \exp(\hat{\Pi}(P^*)) \end{pmatrix} \\ &\quad \times \text{diag} \left( \exp \left\{ -\frac{t}{2} \lambda \right\}, \exp \left\{ \frac{t}{2} \lambda \right\} \right) \\ &:= \hat{\Psi}^\infty(\lambda) \exp T^\infty(\lambda). \end{aligned}$$

From the definition of  $\omega_\infty$ , it follows that

$$\omega_\infty = -\text{Res}_{\lambda=\infty} \text{tr} \hat{\Psi}^\infty(\lambda)^{-1} \frac{\partial}{\partial \lambda} \hat{\Psi}^\infty(\lambda) dT^\infty(\lambda). \quad (5.3)$$

In order to compute  $\omega_\infty$ , we set

$$A(\lambda) = (G^\infty)^{-1} \begin{pmatrix} \hat{\varphi}(P) \exp \hat{\Pi}(P) & \hat{\varphi}(P^*) \exp \hat{\Pi}(P^*) \\ \hat{\psi}(P) \exp \hat{\Pi}(P) & \hat{\psi}(P^*) \exp \hat{\Pi}(P^*) \end{pmatrix}. \quad (5.4)$$

Therefore, we get

$$-\text{tr} \hat{\Psi}^\infty(\lambda)^{-1} \frac{\partial}{\partial \lambda} \hat{\Psi}^\infty(\lambda) dT^\infty(\lambda) = \frac{dt}{2} \lambda \text{tr} A^{-1}(\lambda) A'(\lambda) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad (5.5)$$

and

$$\begin{aligned} & \text{tr} A^{-1}(\lambda) A'(\lambda) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \\ &= \frac{1}{\det \Phi(P)} \left[ \det \begin{pmatrix} \{\hat{\varphi}(P) \exp \hat{\Pi}(P)\}' & \hat{\varphi}(P^*) \exp \hat{\Pi}(P^*) \\ \{\hat{\psi}(P) \exp \hat{\Pi}(P)\}' & \hat{\psi}(P^*) \exp \hat{\Pi}(P^*) \end{pmatrix} \right. \\ & \quad \left. - \det \begin{pmatrix} \hat{\varphi}(P) \exp \hat{\Pi}(P) & \{\hat{\varphi}(P^*) \exp \hat{\Pi}(P^*)\}' \\ \hat{\psi}(P) \exp \hat{\Pi}(P) & \{\hat{\psi}(P^*) \exp \hat{\Pi}(P^*)\}' \end{pmatrix} \right], \end{aligned} \quad (5.6)$$

where  $'$  means the differentiation with respect to the variable  $\lambda$ .

We have normalized the matrix function  $\Psi(\lambda)$  around  $\lambda = \infty$  in Lemma 3.3 and have proved that the monodromy data of  $\Psi(\lambda)$  are independent of  $P_\varphi, P_\psi$  in Theorem 3.4 and its corollary. Therefore, we can choose the parameters,  $P_\varphi, P_\psi$  at our disposal to simplify the calculation.

Firstly, we multiply both the numerators and the denominators of (5.6) by  $\frac{1}{\lambda_\psi - \lambda_\varphi}$ . Then, we take the limit  $P_\psi \rightarrow P_\varphi$  and get

$$\hat{\psi}(P) = \frac{\partial \hat{\varphi}(P)}{\partial \lambda_\varphi}. \quad (5.7)$$

Next, we multiply both the numerators and the denominators of (5.6) by  $\frac{1}{\lambda_\varphi - \lambda}$ . Then, we take the limit  $P_\varphi \rightarrow P$  and obtain

$$\begin{aligned} \text{tr} A^{-1}(\lambda) A'(\lambda) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} &= 2 \frac{1}{\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))} \lim_{P_\varphi \rightarrow P} \frac{\partial}{\partial x_\varphi} \Theta[\mathbf{p}, \mathbf{q}](-U(P) + U(P_\varphi) + \mathbf{v}(t)) \\ & \quad + 2 \frac{\partial}{\partial \lambda} \{\hat{\Pi}(P)\}. \end{aligned} \quad (5.8)$$

From the definition of  $\omega_\infty$ , it follows that

$$\omega_\infty = \frac{dt}{\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))} \sum_{k=1}^g \frac{\partial}{\partial z_k} \{ \Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t)) \} \frac{dU_k}{dx_{\infty^1}}(\infty^1) + \frac{t}{2} dt \left( \frac{1}{6} S(\infty^1) - \frac{W(\infty^1, \infty^2)}{dx_{\infty^1} dx_{\infty^2}} \right).$$

□

## 5.2 Fay's Identities and Rauch's Variational Formulas

In this subsection, we quote Fay's identities and Rauch's variational formulas in order to determine the Hamiltonians  $H_j$  ( $j = 1, 2, \dots, 2g + 2$ ). Lemma 5.2 and 5.3 are devoted to Fay's Identities and Rauch's Variational Formulas, respectively.

**Lemma 5.2.** (1) For  $P, Q \in \mathcal{L}$ ,

$$\frac{\Theta[\mathbf{p}^T, \mathbf{q}^T]^2(U(P) - U(Q))}{\Theta[\mathbf{p}^T, \mathbf{q}^T]^2(0)E^2(P, Q)} = W(P, Q) + \sum_{k,l=1}^g \frac{\partial^2}{\partial z_k \partial z_l} \log \Theta[\mathbf{p}^T, \mathbf{q}^T](0) dU_k(P) dU_l(Q).$$

(2) For  $P, Q \in \mathcal{L}$ ,

$$\begin{aligned} & \frac{\Theta[\mathbf{p}^T, \mathbf{q}^T](2(U(P) - U(Q)))}{\Theta[\mathbf{p}^T, \mathbf{q}^T](0)E^4(P, Q)} - \frac{\Theta[\mathbf{p}^T, \mathbf{q}^T]^4(U(P) - U(Q))}{\Theta[\mathbf{p}^T, \mathbf{q}^T]^4(0)E^4(P, Q)} \\ &= \frac{1}{2} \sum_{k,l,m,n=1}^g \frac{\partial^4}{\partial z_k \partial z_l \partial z_m \partial z_n} \log \Theta[\mathbf{p}^T, \mathbf{q}^T](0) dU_k(P) dU_l(P) dU_m(Q) dU_n(Q). \end{aligned}$$

(3) For  $P, Q \in \mathcal{L}$ ,

$$\begin{aligned} & \frac{\Theta[\mathbf{p}^T, \mathbf{q}^T](0) \Theta[\mathbf{p}, \mathbf{q}](2(U(P) - U(Q)))}{\Theta[\mathbf{p}^T, \mathbf{q}^T](U(P) - (Q))^2 E(P, Q)^2} = W(P, Q) \\ & \quad + \sum_{k,l=1}^g \frac{\partial^2}{\partial z_k \partial z_l} \log \Theta[\mathbf{p}^T, \mathbf{q}^T](U(P) - U(Q)) dU_k(P) dU_l(Q) \\ &= dx_P dx_Q \log \frac{E(P, Q)}{\Theta[\mathbf{p}^T, \mathbf{q}^T](U(P) - U(Q))}, \end{aligned}$$

where  $x_P, x_Q$  are local coordinates of  $P, Q$ , respectively. Especially, since  $\mathcal{L}$  is hyperelliptic, it follows that

$$\frac{\Theta[\mathbf{p}^T, \mathbf{q}^T](0) \Theta[\mathbf{p}, \mathbf{q}](2(U(P) - U(Q)))}{\Theta[\mathbf{p}^T, \mathbf{q}^T](U(P) - (Q))^2 E(P, Q)^2} = dx_P dx_Q \log \frac{1}{m_T(P, Q)}.$$

*Proof.* For (1), (2) and (3), see pp 26, pp 28 and pp 29 in Fay's book [2], respectively.  $\square$

Rauch [10] described the dependence of  $dU_k$  ( $k = 1, 2, \dots, g$ ) and  $\mathbf{B}_{kl}$  ( $k, l = 1, 2, \dots, g$ ) on the moduli of the Riemann surfaces. The moduli space of hyperelliptic curves can be parameterized by the positions of the branch points  $\lambda_j$  ( $j = 1, 2, \dots, 2g + 2$ ). Korotkin [8] proved the variational formulas of the following useful form.

**Lemma 5.3.** (1) For  $P \in \mathcal{L}$ ,

$$\frac{\partial}{\partial \lambda_j} \left\{ \frac{dU_k}{dx_P}(P) \right\} = \frac{1}{2} \frac{W(P, \lambda_j)}{dx_P dx_j} \frac{dU_k}{dx_j}(\lambda_j), \quad (5.9)$$

where  $x_P$  is a local coordinate of  $P$  and  $x_j = \sqrt{\lambda - \lambda_j}$ , which is a local coordinate of the branch point  $\lambda_j$  for any  $j = 1, 2, \dots, 2g + 2$ .

(2) For the branch points  $\lambda_j$  ( $j = 1, 2, \dots, 2g + 2$ ),

$$\frac{\partial \mathbf{B}_{kl}}{\partial \lambda_j} = \pi i \frac{dU_l}{dx_j}(\lambda_j) \frac{dU_k}{dx_j}(\lambda_j) \quad (k, l = 1, 2, \dots, g), \quad (5.10)$$

where  $x_j = \sqrt{\lambda - \lambda_j}$ , which is a local coordinate of  $\lambda_j$ .

### 5.3 The $\tau$ -Function

In this subsection, we compute  $\omega_{\lambda_j}$  ( $j = 1, 2, \dots, 2g + 2$ ) and calculate the  $\tau$ -function.

**Proposition 5.4.** For  $j = 1, 2, \dots, 2g + 2$ ,

$$\begin{aligned} H_j = \frac{\partial}{\partial \lambda_j} \log \Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t)|\mathbf{B}) - \frac{1}{2} \frac{\partial}{\partial \lambda_j} \log \det \mathcal{A} - \frac{1}{8} \frac{\partial}{\partial \lambda_j} \log \prod_{k < l} (\lambda_k - \lambda_l) \\ + \frac{\partial}{\partial \lambda_j} \left\{ \frac{t^2}{4} \left( \frac{1}{6} S(\infty^1) - \frac{W(\infty^1, \infty^2)}{dx_{\infty^1} dx_{\infty^2}} \right) \right\}. \end{aligned}$$

*Proof.* By direct calculation, we obtain

$$\frac{1}{2} \operatorname{tr} (\Psi'(\lambda) \Psi^{-1}(\lambda))^2 = -\frac{\det(\Phi_x)}{\det \Phi} + \frac{1}{4} \left( \frac{(\det \Phi)_\lambda}{\det \Phi} \right)^2.$$

We calculate  $\omega_{\lambda_j}$  in the same way as  $\omega_\infty$  in Proposition 5.1.

We multiply both the numerators and the denominators of  $\frac{\det(\Phi_\lambda)}{\det \Phi}, \frac{(\det \Phi)_\lambda}{\det \Phi}$  by  $\frac{1}{\lambda_\varphi - \lambda_\psi}$ . Then, we take the limit  $P_\psi \rightarrow P_\varphi$  and get

$$\psi(P) = \frac{\partial \varphi(P)}{\partial \lambda_\varphi}. \quad (5.11)$$

Furthermore, we multiply both the numerators and the denominators of  $\frac{\det(\Phi_\lambda)}{\det \Phi}$ ,  $\frac{(\det \Phi)_\lambda}{\det \Phi}$  by  $\frac{1}{\lambda_\varphi - \lambda}$  and take the limit  $P_\varphi \rightarrow P$ . Then, we obtain

$$\begin{aligned} \frac{(\det \Phi)_\lambda}{\det \Phi} &= 2 \frac{\partial}{\partial \lambda} \log \Theta[\mathbf{p}^S, \mathbf{q}^S](-2U(P)), \\ \frac{\det(\Phi_\lambda)}{\det \Phi} &= \frac{1}{\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))} \lim_{P_\varphi \rightarrow P} \frac{\partial^2}{\partial \lambda \partial \lambda_\varphi} \Theta[\mathbf{p}, \mathbf{q}](-U(P) + U(P_\varphi) + \mathbf{v}(t)) \\ &\quad + \frac{2}{\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))} \lim_{P_\varphi \rightarrow P} \frac{\partial}{\partial \lambda} \Theta[\mathbf{p}, \mathbf{q}](-U(P) + U(P_\varphi) + \mathbf{v}(t)) \frac{\partial}{\partial \lambda} \Pi(P) \\ &\quad + \frac{1}{\Theta[\mathbf{p}^S, \mathbf{q}^S](-2U(P))} \lim_{P_\varphi \rightarrow P} \frac{\partial^2}{\partial \lambda \partial \lambda_\varphi} \Theta[\mathbf{p}^S, \mathbf{q}^S](-U(P) - U(P_\varphi)) \\ &\quad - \left( \frac{\partial}{\partial \lambda} \Pi(P) \right)^2, \end{aligned}$$

which implies that

$$\frac{1}{2} \operatorname{tr} (\Psi'(\lambda) \Psi^{-1}(\lambda))^2 = - \frac{\partial^2}{\partial \lambda \partial \lambda_\varphi} \log \Theta[\mathbf{p}^S, \mathbf{q}^S](-U(P) - U(P_\varphi)) \Big|_{P_\varphi=P} \quad (5.12)$$

$$- \frac{1}{\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))} \frac{\partial^2}{\partial \lambda \partial \lambda_\varphi} \Theta[\mathbf{p}, \mathbf{q}](-U(P) + U(P_\varphi) + \mathbf{v}(t)) \Big|_{P_\varphi=P} \quad (5.13)$$

$$- \frac{2}{\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))} \frac{\partial}{\partial \lambda} \Theta[\mathbf{p}, \mathbf{q}](-U(P) + U(P_\varphi) + \mathbf{v}(t)) \frac{\partial}{\partial \lambda} \Pi(P) \Big|_{P_\varphi=P} \quad (5.14)$$

$$+ \left( \frac{\partial}{\partial \lambda} \Pi(P) \right)^2. \quad (5.15)$$

Firstly, we calculate the residue of (5.12) at  $\lambda = \lambda_j$ . For the purpose, we set

$$P = \lambda_j, \quad x_P = x_j := \sqrt{\lambda - \lambda_j}$$

in (3.7) of Lemma 3.2. Then, we have

$$\begin{aligned} \operatorname{Res}_{\lambda=\lambda_1} &\left\{ - \frac{\partial^2}{\partial \lambda \partial \lambda_\varphi} \log \Theta[\mathbf{p}^S, \mathbf{q}^S](-U(P) - U(P_\varphi)) \Big|_{P_\varphi=P} \right\} \\ &= \frac{1}{8} \sum_{k \neq j} \frac{n_j n_k}{\lambda_j - \lambda_k} - \frac{1}{4 \Theta[\mathbf{p}^T, \mathbf{q}^T]}(0) \sum_{l,k=1}^g \frac{\partial^2 \Theta[\mathbf{p}^T, \mathbf{q}^T]}{\partial z_l \partial z_k}(0) \frac{dU_l}{dx_j}(\lambda_j) \frac{dU_k}{dx_j}(\lambda_j), \end{aligned}$$

where  $n_k = 1$  for  $\lambda_k \in T$  and  $n_k = -1$  for  $\lambda_k \notin T$ . For the calculation, we use (5.10) in Lemma 5.3 and the heat equation

$$\frac{\partial^2 \Theta[\mathbf{p}, \mathbf{q}](\mathbf{z}|\mathbf{B})}{\partial z_l \partial z_k} = 4\pi i \frac{\partial \Theta[\mathbf{p}, \mathbf{q}](\mathbf{z}|\mathbf{B})}{\partial \mathbf{B}_{lk}}. \quad (5.16)$$

Then, we get

$$\begin{aligned} \text{Res}_{\lambda=\lambda_j} \left\{ -\frac{\partial^2}{\partial \lambda \partial \lambda_\varphi} \log \Theta[\mathbf{p}^S, \mathbf{q}^S](-U(P) - U(P_\varphi)) \Big|_{P_\varphi=P} \right\} \\ = \frac{1}{8} \sum_{k \neq j} \frac{n_j n_k}{\lambda_j - \lambda_k} - \frac{\partial}{\partial \lambda_j} \log \Theta[\mathbf{p}^T, \mathbf{q}^T](0|\mathbf{B}). \end{aligned}$$

By using the Thomae's formula

$$\Theta^4[\mathbf{p}^T, \mathbf{q}^T](0) = \pm \frac{(\det \mathcal{A})^2}{(2\pi i)^{2g}} \prod_{l < k, l, k=1}^{g+1} (\lambda_{i_l} - \lambda_{i_k}) \prod_{l < k, l, k=1}^{g+1} (\lambda_{j_l} - \lambda_{j_k}),$$

we get

$$\begin{aligned} \text{Res}_{\lambda=\lambda_j} \left\{ -\frac{\partial^2}{\partial \lambda \partial \lambda_\varphi} \log \Theta[\mathbf{p}^S, \mathbf{q}^S](-U(P) - U(P_\varphi)) \Big|_{P_\varphi=P} \right\} \\ = -\frac{1}{2} \frac{\partial}{\partial \lambda_j} \log \det \mathcal{A} - \frac{1}{8} \frac{\partial}{\partial \lambda_j} \log \prod_{k < l} (\lambda_k - \lambda_l). \end{aligned}$$

Secondly, we calculate the residue of the sum of (5.13) and (5.14) at  $\lambda = \lambda_j$ , which is

$$\begin{aligned} \frac{1}{4\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))} \sum_{k, l=1}^g \frac{\partial^2 \Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))}{\partial z_k \partial z_l} \frac{dU_k(\lambda_j)}{dx_j} \frac{dU_l(\lambda_j)}{dx_j} \\ + \frac{1}{2\Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))} \sum_{k=1}^g \frac{\partial \Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t))}{\partial z_k} \times t \frac{W(\lambda_1, \infty^1)}{dx_j dx_{\infty^1}}. \quad (5.17) \end{aligned}$$

From Lemma 5.3, it follows that (5.17) is

$$\frac{\partial}{\partial \lambda_j} \log \Theta[\mathbf{p}, \mathbf{q}](\mathbf{v}(t)|\mathbf{B}).$$

Lastly, we deal with the residue of (5.15) at  $\lambda = \lambda_j$ , which is  $\frac{t^2}{4} \left( \frac{W(\lambda_j, \infty^1)}{dx_j dx_{\infty^1}} \right)^2$ . We prove that if  $\lambda_j \in T$ ,

$$\frac{\partial}{\partial \lambda_j} \left\{ \frac{t^2}{4} \left( \frac{1}{6} S(\infty^1) - \frac{W(\infty^1, \infty^2)}{dx_{\infty^1} dx_{\infty^2}} \right) \right\} = \frac{t^2}{4} \left( \frac{W(\lambda_j, \infty^1)}{dx_j dx_{\infty^1}} \right)^2.$$

If  $\lambda_j \notin T$ , this formula can be proved in the same way.

From Lemma 3.2 and Lemma 5.2 (1), it follows that

$$\begin{aligned}
\frac{1}{6}S(\infty^1) - \frac{W(\infty^1, \infty^2)}{dx_{\infty^1}dx_{\infty^2}} &= \frac{1}{16} \left( \sum_{k=1}^{g+1} \lambda_{i_k} - \sum_{k=1}^{g+1} \lambda_{j_k} \right)^2 - \frac{\Theta[\mathbf{p}^T, \mathbf{q}^T]^2 (U(\infty^1) - U(\infty^2))}{\Theta[\mathbf{p}^T, \mathbf{q}^T]^2(0) \{E(\infty^1, \infty^2) \sqrt{dx_{\infty^1}} \sqrt{dx_{\infty^2}}\}^2} \\
&\quad - 2 \sum_{k,l=1}^g \frac{\partial^2}{\partial z_k \partial z_l} \log \Theta[\mathbf{p}^T, \mathbf{q}^T](0) \frac{dU_k}{dx_{\infty^1}}(\infty^1) \frac{dU_l}{dx_{\infty^1}}(\infty^1) \\
&= \frac{1}{8} \left( \sum_{k=1}^{g+1} \lambda_{i_k} - \sum_{k=1}^{g+1} \lambda_{j_k} \right)^2 \\
&\quad - 2 \sum_{i,j=1}^g \frac{\partial^2}{\partial z_i \partial z_j} \log \Theta[\mathbf{p}^T, \mathbf{q}^T](0) \frac{dU_i}{dx_{\infty^1}}(\infty^1) \frac{dU_j}{dx_{\infty^1}}(\infty^1),
\end{aligned}$$

where we have used

$$\frac{\Theta[\mathbf{p}^T, \mathbf{q}^T](U(\infty^1) - U(\infty^2))}{\Theta[\mathbf{p}^T, \mathbf{q}^T](0)E(\infty^1, \infty^2)} = m_T(\infty^1, \infty^2) = \frac{\sqrt{-1}}{4} \left( \sum_{k=1}^{g+1} \lambda_{i_k} - \sum_{k=1}^{g+1} \lambda_{j_k} \right) \sqrt{dx_{\infty^1}} \sqrt{dx_{\infty^2}}.$$

Then, from Lemma 5.3, it follows that

$$\begin{aligned}
&\frac{\partial}{\partial \lambda_j} \left\{ \frac{1}{6}S(\infty^1) - \frac{W(\infty^1, \infty^2)}{dx_{\infty^1}dx_{\infty^2}} \right\} \\
&= \frac{1}{4} \left( \sum_{k=1}^{g+1} \lambda_{i_k} - \sum_{k=1}^{g+1} \lambda_{j_k} \right) \\
&\quad - \frac{1}{2} \sum_{k,l,m,n=1}^g \frac{\partial^4}{\partial z_k \partial z_l \partial z_m \partial z_n} \log \Theta[\mathbf{p}^T, \mathbf{q}^T](0) \frac{dU_k}{dx_{\infty^1}}(\infty^1) \frac{dU_l}{dx_{\infty^1}}(\infty^1) \frac{dU_m}{dx_{\infty^1}}(\infty^1) \frac{dU_n}{dx_{\infty^1}}(\infty^1) \\
&\quad - \left( \sum_{k,l=1}^g \frac{\partial^2}{\partial z_k \partial z_l} \log \Theta[\mathbf{p}^T, \mathbf{q}^T](0) \frac{dU_k}{dx_j}(\lambda_j) \frac{dU_l}{dx_{\infty^1}}(\infty^1) \right)^2 \\
&\quad - 2 \left( \sum_{k,l=1}^g \frac{\partial^2}{\partial z_k \partial z_l} \log \Theta[\mathbf{p}^T, \mathbf{q}^T](0) \frac{dU_k}{dx_j}(\lambda_j) \frac{dU_l}{dx_{\infty^1}}(\infty^1) \right) \frac{W(\lambda_j, \infty^1)}{dx_j dx_{\infty^1}}.
\end{aligned}$$

Thus, from Lemma 5.2 (1) and (2), we have

$$\begin{aligned} \frac{\partial}{\partial \lambda_1} \left\{ \frac{1}{6} S(\infty^1) - \frac{W(\infty^1, \infty^2)}{dx_{\infty^1} dx_{\infty^2}} \right\} &= \frac{1}{4} \left( \sum_{k=1}^{g+1} \lambda_{i_k} - \sum_{k=1}^{g+1} \lambda_{j_k} \right) - \frac{\Theta[\mathbf{p}^T, \mathbf{q}^T] (2(U(\infty^1) - U(\lambda_j)))}{\Theta[\mathbf{p}^T, \mathbf{q}^T](0) \{E(\lambda_j, \infty^1) \sqrt{dx_j} \sqrt{dx_{\infty^1}}\}^4} \\ &\quad + \left( \frac{W(\lambda_j, \infty^1)}{dx_j dx_{\infty^1}} \right)^2. \end{aligned}$$

Therefore, from Lemma 5.2 (3), we obtain

$$\frac{\partial}{\partial \lambda_j} \left\{ \frac{1}{6} S(\infty^1) - \frac{W(\infty^1, \infty^2)}{dx_{\infty^1} dx_{\infty^2}} \right\} = \left( \frac{W(\lambda_j, \infty^1)}{dx_j dx_{\infty^1}} \right)^2,$$

which completes the proof of the proposition. □

Since we have calculated the Hamiltonians  $H_t, H_1, H_2, \dots, H_{2g+2}$ , we finally obtain Theorem 0.1.

## References

- [1] P. Deift., A. Its., A., Kapaev. and X. Zhou., *On the algebro-geometric integration of the Schlesinger equations and on elliptic solutions of the Painlevé 6 equation*, *Commun. Math. Phys.* **203**. (1999) 613-633.
- [2] J. D. Fay., *Theta-functions on Riemann surfaces*, *Lecture Notes in Mathematics*, Vol. **352**, Springer-Verlag, Berlin-New York, (1973).
- [3] H. Farkas., I. Kra., *Riemann Surfaces*, 2nd ed, Springer.
- [4] K. Matsuda., *Isomonodromic deformation with an irregular singularity and the elliptic theta functions*, *J. Phys. A: Math. Theor.* **40**. (2007) 11939-11960.
- [5] M. Jimbo, T. Miwa and K. Ueno, *Monodromy preserving deformations of linear differential equations with rational coefficients*, *Physica* **1D** (1980) 80-158.
- [6] M. Jimbo and T. Miwa, *Monodromy preserving deformations of linear differential equations with rational coefficients II*, *Physica* **2D** (1981) 407-448.
- [7] A. V. Kitaev and D. A. Korotkin, *On solutions of the Schlesinger equations in terms of  $\Theta$  functions*, *Intern. Math. Research Notices* **17** (1998) 877-905.



- [8] D. A. Korotkin, *Solution of matrix Riemann-Hilbert problems with quasi-permutation monodromy matrices*, *Math. Ann.* **329**, no. 2, (2004), 335–364.
- [9] D. Mumford, *Tata Lectures on Theta I*, *Progress in Math*, v 28, Birkhauser (1983).
- [10] H. E. Rauch., *Weierstrass points, branch points and moduli of Riemann Surfaces*, *Comm. Pure. Appl. Math.* **12** (1959) 543-560.
- [11] L. Schlesinger., *Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten*, *J. Reine u. Angew. Math.* **141** (1912) 96-145.

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